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Dilute-crack approximations for damage and fracture in disordered structures

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Abstract. Quenched disorder in local breaking thresholds within a two-dimensional lattice is studied, using dilute-crack approximations. Two different failure criteria are compared. Application of classical theory of strength, instead of fracture mechanics energy balance criterion, yields two times greater critical crack length, and two times greater amount of damage, in the case of failure threshold distributions without nonzero lower cutoff. Upper bounds of threshold distributions chosen correspondingly, critical voltages become similar. Normalized distribution of crack sizes appears to be independent of the applied mesoscopic failure criterion, as well as of the details of the distribution of the values for the failure threshold.

PACS. 62.20.Mk Fatigue, brittleness, fracture, and cracks

1 Introduction

The effect of the size of a system on the strength of the system often is logarithmic. The same applies to the size of the largest crack the system may contain. In the case of a dilute network of dimensionality d, linear size L, and void fraction q, the size of the largest void can be deduced from the fact that there is only one largest void. The appearance probability of a row of r vacancies is q^r, and $L^d q^{r_{max}} = 1$. Then, the size of the largest void is $r_{\max} = -\frac{d \ln L}{\ln q}$ [1,2]. Further, there is a stress concentration at the tip of the crack, proportional to $1 + k r_{max}^m$, where k and m are constants, and correspondingly, the size effect on critical stress is

$$\frac{\sigma_c(q)}{\sigma_c(0)} \approx \frac{1}{1 + k r_{max}^m} = \frac{1}{1 - k \left(\frac{d \ln L}{\ln q} \right)^m}. \qquad (1)$$

Instead of dilution disorder, we are here interested in fracture and damage in disordered systems with continuous distributions of local failure thresholds. In particular, we restrict ourselves to quenched disorder in local breaking thresholds within a two-dimensional square lattice. Such a lattice, here discussed as an electrical conductivity problem, is a representation of a sheet material with disorder on the mesoscopic level – a size scale much greater than the molecular size, but small in relation to the size of any piece of material used in technical applications.

We will start our discussion by reviewing different failure theories. Then, we will apply the classical theory of strength, as well as the energy balance failure criterion, on the mesoscopic level. We will develop dilute-crack approximations for the critical crack length, critical voltage, crack appearance probability, crack size distribution, and damage. Results for the two kinds of failure criteria will be compared.

2 Alternative failure theories, and energy balance failure criterion on a mesoscopic level

Ancient constructions manifest knowledge of mankind, regarding strength and fracture of structures. Galileo, at 1638, hardly was the first to formulate the concepts of stress and momentum, as well as the associated scaling laws. The classical theory of strength thus obviously is of ancient origin.

The origin of the statistical theory of strength is much less obvious. Sixteenth century, Leonardo observed the strength of wires to be inversely proportional to length. His findings were confronted by Galileo, but later elaborated by Mariotte (1686), who associated the statistical size effect to the existence of flaws. Finally, Weibull presented his celebrated semi-empirical weakest-link theory in 1939 [3].

A completely different approach was presented by Griffith at 1920 [4]. An ideal elastic-brittle body was considered as static and thermodynamically reversible. New crack surface may form only if crack propagation releases potential energy enough to overcome an energy requirement for the creation of the new crack surface. A large plate with a central through crack of length $2a$, having a potential energy Π and total crack surface energy $2\gamma B 2a$, where γ is specific surface energy and B is thickness, the

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first law of thermodynamics requires $\frac{d\Pi}{d2a} + \frac{d2\gamma B2a}{d2a} \leqslant 0$. The release rate of potential energy can be computed as $\frac{d\Pi}{d2a} = -\frac{B\sigma^2\pi a}{E}$, where σ is nominal stress, and E is elastic material constant. The critical value of the potential energy release rate satisfies the equality $\left(\frac{d\Pi}{d2a}\right)_c + \frac{d2\gamma B2a}{d2a} = 2\gamma - \frac{\sigma_c^2\pi a_c}{E} = 0$. Thus, the specific surface energy can be estimated on the basis of the critical nominal stress, critical crack length, and elastic constant, or the critical nominal stress can be predicted on the basis of the specific surface energy, crack length, and elastic constant.

Within the framework of Linear Elastic Fracture Mechanics, the energy release rate as a failure criterion can be expressed in terms of a stress intensity factor and elastic stiffness [5]. It is worth noting that the stress intensity factor — or fracture toughness — describing singular stress fields in the vicinity of a sharp crack, it does not have any direct connection to the classical theory of strength, or to any other argument applying a finite stress or strain level as a local failure criterion. Rather, it is just another way to express the energy release rate argument.

Most materials are not ideally elastic-brittle, but they suffer plastic deformations. The work consumed in such plastic deformations may be much greater than the surface energy [6–9]. However, in configurations where the plastic dissipation is associated with the crack propagation, the surface energy 2γ can be replaced with the critical value of the total energy release rate (per thickness unit):

$$\left(\frac{d\Pi}{d2a}\right)_c = G_c = \frac{\sigma_c^2\pi a_c}{E}. \tag{2}$$

The fracture resistance G_c — or the critical value of the J-integral — may constitute energy consumed for the creation of damage within an eventual damage band, in addition to plastic yielding and surface energy. It may change as a function of crack advance. However, as a simplification, this quantity, also called the Specific Essential Work of Fracture, can be taken as a property characteristic to any material.

Thus we have two kinds of failure criteria under discussion. The classical theory of strength favors the critical stress. Applying Hooke's law, it may be equivalently expressed in terms of critical strain, or the critical amount of work under the stress-strain–curve, often denoted as toughness (not to be mixed with Fracture Toughness). On the other hand, any of the above may be replaced by the energy balance failure criterion, the critical energy release rate.

Elements which do not contain sharp cracks or defects can be assumed to follow the classical theory. On the other hand, elements containing sharp cracks or other stress-enhancing defects may not be reasonably described by the locally applied classical theory of strength. A continuum approximation for the stress field in the vicinity of a sharp crack yields a singularity, and even if crack tip curvature radius is limited to the molecular size scale, the resulting stress at the crack tip becomes very high. Then, element failure becomes determined by whether or not the energy balance criterion for fracture becomes fulfilled. In other words, the brittleness of the mesoscopic elements determines whether a theory of strength or a theory of fracture energy balance is more appropriate. Surprisingly, no statistical treatment conducted so far seems to apply the energy balance criterion on mesoscopic material elements.

3 Power-law distributed classical failure thresholds

Let us study a network of fuses with unit conductance, with power-law-distributed failure thresholds. Any failure threshold x is assumed to vary between zero and a maximum x_+, and to have a probability density function

$$\frac{(1-\beta)}{x_+^{1-\beta}} x^{-\beta}, \tag{3}$$

where $\beta < 1$.

Applying the classical theory of strength on the mesoscopic level, the probability of a fuse to burn at a local current v_l is

$$p(v_l) = \int_0^{v_l} \frac{(1-\beta)}{v_+^{1-\beta}} x^{-\beta} dx = \left(\frac{v_l}{v_+}\right)^{1-\beta}. \tag{4}$$

At the edge of an existing crack of length r, the local current is

$$v_r = \alpha_r v \approx \left(1 + c\sqrt{r}\right) v, \tag{5}$$

where c is a constant, and v is the external voltage applied per unit length [2,10].

The probability of burning a fuse at the edge of an existing crack of length r is, in particular in the case of a long crack,

$$p_r = \left(\frac{v_r}{v_+}\right)^{1-\beta} \approx \left(c\sqrt{r}\frac{v}{v_+}\right)^{1-\beta} = r^{\frac{1-\beta}{2}} \left(c\frac{v}{v_+}\right)^{1-\beta}. \tag{6}$$

The probability of a linear crack of length $r+1$ to exist at any particular site is

$$P_{r+1} = p_0 p_1 p_2 ... p_r \left(1 - p_{r+1}\right)^2. \tag{7}$$

The logarithm of P_r can be computed as an integral for large r. The result is

$$\ln P_r \approx (1-\beta)\left(\ln(c\frac{v}{v_+}) - \frac{1}{2} + \frac{1}{2}\ln r\right) r. \tag{8}$$

Thus

$$P_r \approx \left(\frac{cv\sqrt{r}}{v_+\sqrt{e}}\right)^{(1-\beta)r} = \left(\frac{v_r}{v_+\sqrt{e}}\right)^{(1-\beta)r}. \tag{9}$$

Such a crack becomes unstable at critical applied voltage v_c and critical length of the largest crack $r_{\max c}$, where

$v_{r_{max}} \approx v_c c\sqrt{r_{maxC}} \approx v_+$. Thus the average voltage drop per unit length at instability is

$$v_c \approx \frac{v_+}{c\sqrt{r_{maxC}}}. \tag{10}$$

We find that the treatment agrees with the classical Fracture Mechanics crack length effect on critical nominal stress. On the other hand, the total number of cracks of length r is $N(r) = L^2 P_r$. Setting this number to unity at critical nominal stress yields the size of the longest crack at the onset of instability. It is

$$r_{maxC} = \frac{4 \ln L}{1 - \beta}. \tag{11}$$

Now, the critical crack length hardly can be greater than the linear size of the system. Thus we have to require that

$$\beta < 1 - \frac{4 \ln L}{L}. \tag{12}$$

Substituting equation (11) into (10) yields the critical nominal stress (or critical applied voltage)

$$v_c \approx \frac{v_+}{c}\sqrt{\frac{1-\beta}{4 \ln L}}. \tag{13}$$

Now, substituting $v = \overline{v} v_c$ and $r = \overline{r} r_{maxC}$ into equation (9) yields the crack existence probability for any relative crack size at any relative voltage, in relation to the critical values. The result is

$$P_r \approx \left(\frac{\overline{v}\sqrt{\overline{r}}}{\sqrt{e}} \right)^{4\overline{r} \ln L} = \left(\frac{\overline{v}^2 \overline{r}}{e} \right)^{2\overline{r} \ln L}. \tag{14}$$

It appears that the crack appearance probability, as a function of relative crack size, relative voltage and system size, is independent on the upper limit of failure threshold distribution. More remarkably, the crack appearance probability also is independent of the distribution parameter β!

4 Power-law distributed energetic failure thresholds

Let us still discuss a network of fuses with unit conductance. In the random fuse network, the quantity analogous to energy release rate is the differential electrical power resulting from burning of a fuse. Instead of the breaking current, the critical value of the power differential is now the failure criterion for any fuse: $g_i = -\left(\sum_k v_k \Delta v_{ki} \right)_c$, where v_k is voltage drop over fuse k, and Δv_{ki} is change in voltage drop over fuse k due to burning of fuse i.

The power differential resulting from burning of a fuse strongly depends on local geometry. Transmission power is reduced in fuses adjacent to the burning fuse in the direction of the main current, but there is a power enhancement in fuses laterally adjacent to the burning fuse. We are roughly approximating the power differential for the formation of a new single-fuse crack as

$$\sum_k v_k \Delta v_{ki} = -3v^2 + 2(\alpha - 1) v^2, \tag{15}$$

where the first term accounts for differential power within the burning fuse and two fuses adjacent to it in the direction of the main current, and the second term for power enhancement in the two along-current aligned fuses adjacent to the burning fuse in the lateral direction. In the case of a fuse burning at the tip of an existing crack of length r, we are approximating the power differential as

$$\sum_k v_k \Delta v_{ki} = -(2r + 3) v^2 + (\alpha_r - 1) v^2, \tag{16}$$

where α_r at the limit of large r is given in equation (5). This being proportional to the square root of n, the latter term of equation (16) may be neglected at large r.

Now, the probability of a fuse to burn at a local power differential g_l is

$$p(g_l) = \int_0^{g_l} \frac{(1-\beta)}{g_+} x^{-\beta} dx = \left(\frac{g_l}{g_+} \right)^{1-\beta}. \tag{17}$$

The probability of burning a fuse at the edge of an existing crack of length r is

$$p_r = \left(\frac{g_r}{g_+} \right)^{1-\beta} \approx \left(\frac{2rv^2}{g_+} \right)^{1-\beta} = \left(\frac{2r}{g_+} \right)^{1-\beta} v^{2(1-\beta)}, \tag{18}$$

The logarithm of P_r can be computed as an integral for large r. The result is

$$\ln P_r \approx (1-\beta)(\ln 2 - \ln g_+ + 2 \ln v - 1 + \ln r) r. \tag{19}$$

Thus

$$P_r \approx \left(\frac{2v^2 r}{g_+ e} \right)^{(1-\beta)r} = \left(\frac{g_r}{g_+ e} \right)^{(1-\beta)r}. \tag{20}$$

Such a crack becomes unstable when $g_{r_{max}} \approx 2 r_{maxC} v_c^2 \approx g_+$. Thus the average voltage drop per unit length at instability is

$$v_c \approx \sqrt{\frac{g_+}{2 r_{maxC}}}. \tag{21}$$

Again, the result agrees with classical Fracture Mechanics. The total number of cracks of length r is $N(r) = L^2 P_r$. Setting this number to unity at critical nominal stress yields the size of the longest crack at the onset of instability. It is

$$r_{maxC} = \frac{2 \ln L}{1 - \beta}. \tag{22}$$

Interestingly, we find that the critical crack length is exactly half of the critical crack length in the case of mesoscopically applied classical theory of strength. Now, the critical crack length hardly can be greater than the linear size of the system. Thus we have to require that

$$\beta < 1 - \frac{2 \ln L}{L}. \tag{23}$$

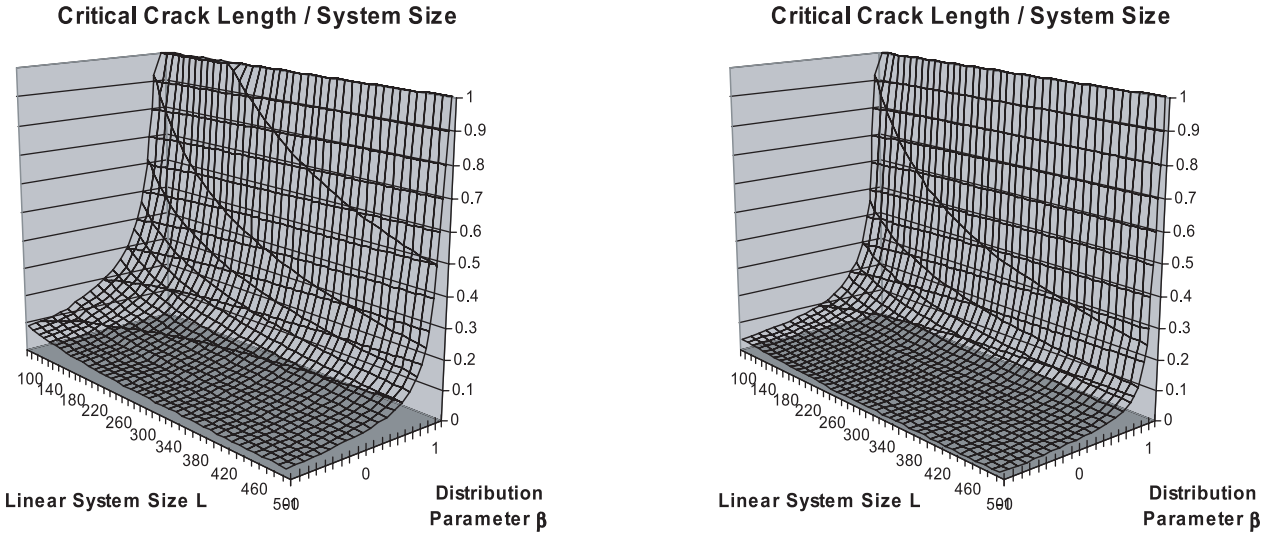

Fig. 1. Critical crack length in relation to linear system size, as a function of linear system size and failure threshold distribution parameter, according to the dilute-crack approximation. The distribution parameter β runs from -1 to the vicinity of 1. On the left, local failure thresholds according to the Classical Theory of Strength. On the right, failure thresholds according to the Energy Balance Criterion.

Substituting equation (22) into (21) yields the critical nominal stress (or critical applied voltage)

$$v_c \approx \sqrt{\frac{g_+ (1-\beta)}{4 \ln L}}. \tag{24}$$

Now, substituting $v = \overline{v} v_c$ and $r = \overline{r} r_{maxC}$ into equation (20) yields the crack existence probability for any relative crack size at any relative voltage. The result is

$$P_r \approx \left(\frac{\overline{v}^2 \overline{r}}{e} \right)^{2\overline{r} \ln L}. \tag{25}$$

Surprisingly, we find that equation (25) equals equation (14). Thus, the crack appearance probability, as a function of relative crack size and relative voltage, as well as system size, is independent of the failure criterion! Again, the crack appearance probability, as a function of relative crack size, relative voltage and system size, is independent of the distribution parameter β.

Figure 1 shows the critical crack length as a function of the distribution parameter and system size. We find that once the distribution parameter approaches unity, the critical crack size approaches system size. The distribution parameter getting smaller, the critical crack size vanishes, negative infinity in β corresponding to the absence of disorder in the failure thresholds. The classical theory of strength yields critical crack lengths exactly two times those of the energy balance criterion.

Figure 2 shows the average critical voltage drop as a function of the distribution parameter and system size. We find that once the distribution parameter approaches unity, the critical voltage drop approaches zero. The distribution parameter getting smaller, the critical crack size vanishes, negative infinity in β corresponding to the absence of disorder in the failure thresholds. Equations (13) and (24) being based on dilute-crack approximations for long cracks, their validity ceases with vanishing crack size. Negative infinity in β accumulates the failure threshold probability mass in the vicinity of the upper limit of the distribution, and with vanishing crack size, the critical voltage drop tends to the upper limit. Results for the two different kinds of failure criteria are equal to each other, provided $\sqrt{g_+} = \frac{v_+}{c}$.

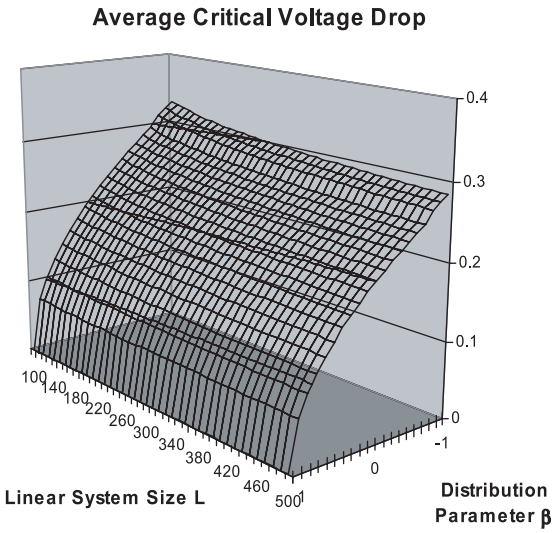

Fig. 2. Average critical voltage drop, in relation to linear system size, as a function of linear system size and failure threshold distribution parameter, according to the dilute-crack approximation. The distribution parameter β runs from -1 to the vicinity of 1. $\sqrt{g_+} = \frac{v_+}{c} = 1$.

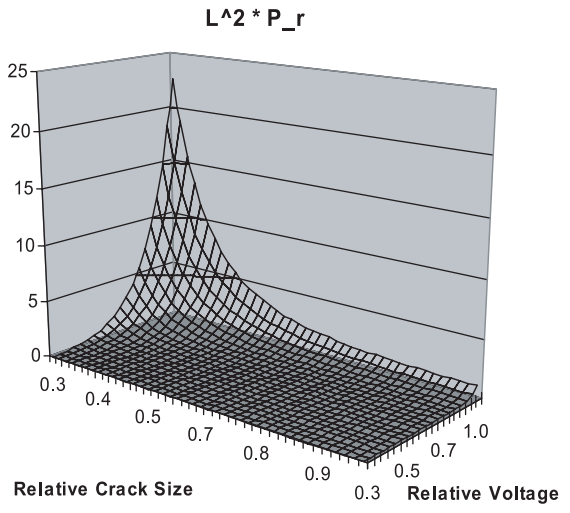

Fig. 3. Average number of cracks of relative size $\overline{r}$ within a lattice of linear size 100, as a function of relative crack size and relative applied voltage, according to the dilute-crack approximation. The result is independent of the distribution parameter β, and equal for both of the applied failure criteria.

We find from equations (14) and (25), that within the dilute-crack approximation, using a power-law distribution of the failure thresholds, the probability of a crack of size r to exist at any site on the lattice P_r is a function of system size L, relative crack size $\overline{r} = \frac{r}{r_{maxC}}$ and relative applied voltage $\overline{v} = \frac{v}{v_c}$. The average number of cracks of size r being $L^2 P_r$, the number of cracks of relative size $\overline{r}$ within a lattice of size 100 is shown in Figure 3. The crack appearance probability is independent of the distribution parameter β, and it is the same for both of the applied failure criteria.

The amount of damage in cracks of relative size $\overline{r}$, as a function of relative crack size and relative applied voltage, is achieved from the distribution shown in Figure 3 by multiplying any crack appearance probability by crack size. The crack size, in turn, can be expressed as the product of relative crack size, which appears in Figure 3, and the critical crack size. The critical crack size does depend on the applied failure criterion, as well as on the distribution parameter β, according to equations (11) and (22), and according to Figure 1. The dependency on the failure criterion, however, is particularly simple: the critical crack size in the case of the classical theory of strength is exactly two times the one resulting from the energy balance argument. The normalized distribution of crack sizes being similar, this also means that the extent of damage is exactly two times greater at any applied relative voltage.

5 Conclusions

An observation rather surprising for the author is that the normalized distribution of crack sizes, as a function of the relative voltage, appears to be independent of the applied mesoscopic failure criterion, as well as the details of the distribution of the values for the failure threshold (Fig. 3). It appears that the similarity applies when the failure threshold distribution extends all the way to zero. It can be verified numerically that the similarity does not extend to distributions with a nonzero lower cutoff.

In the case of failure threshold distributions including zero, the critical crack size is, in the case of the classical theory of strength, exactly two times the one resulting from the energy balance argument (Eqs. (11) and (22)). The normalized distribution of crack sizes being similar, this also means that the extent of damage is exactly two times greater at any applied relative voltage. The critical crack size always is proportional to the logarithm of linear system size, which interestingly agrees with dilute-system results [1,2].

The critical applied voltage is proportional to the upper bound of the failure threshold distribution (Eqs. (10) and (21)). Within the classical theory of strength, the critical applied voltage is inversely proportional to the square root of the critical crack size, corresponding to the classical fracture mechanics result (Eq. (10)). Applying the energy balance criterion, the critical applied voltage is inversely proportional to the square root of *two times* the critical crack size (Eq. (21)). On the other hand, the critical crack size is two times greater in the case of the classical theory of strength. Thus, upper bounds of the failure threshold distributions chosen correspondingly, the critical voltage is independent of the failure criterion, regardless the difference in critical crack size, and the corresponding difference in the extent of damage. Interestingly, considering that the present analysis has been implemented for large cracks, the size effect on critical nominal stress agrees with that of dilatation disorder (Eq. (1)) [1,2].

It would be of interest to generalize the treatment for a failure criterion x, where the dependency of the local value of the failure parameter on local crack length r and applied voltage v is $x_r = cv^\gamma r^\alpha$.

In the case of an exponential distribution without a lower nonzero cutoff the probability of a fuse to burn at local failure parameter value is

$$p(x_r) = \int_0^{x_r} \frac{(1-\beta)\, x^{-\beta}}{x_+^{1-\beta}} dx = \left(\frac{x_r}{x_+} \right)^{1-\beta} = \left(\frac{cv^\gamma r^\alpha}{x_+} \right)^{1-\beta} . \tag{26}$$

Computing the logarithm of the probability of a linear crack to exist at any site P_r as an integral reveals

$$P_r \approx \left(\frac{cv^\gamma r^\alpha}{x_+ e^\alpha} \right)^{(1-\beta)r} = \left(\frac{c\,(\overline{v} v_c)^\gamma\, (\overline{r} r_c)^\alpha}{x_+ e^\alpha} \right)^{(1-\beta)r} . \tag{27}$$

The critical applied voltage is

$$v_c = \left(\frac{x_+}{c r_c^\alpha} \right)^{\frac{1}{\gamma}} . \tag{28}$$

Considering that there is only one critical crack, its length is

$$r_c = \frac{2 \ln L}{\alpha\,(1-\beta)} . \tag{29}$$

Thus we find that equation (11) is a special case where $\alpha = \frac{1}{2}$, and equation (22) is a special case where $\alpha = 1$. Substituting equations (28) and (29) into (27) gives

$$P_r \approx \left(\frac{(\overline{v})^{\frac{\gamma}{\alpha}} \overline{r}}{e} \right)^{\overline{r} \ln L} . \quad (30)$$

Thus we find that in the case of an exponential distribution without nonzero lower cutoff, the crack size distribution as a function of relative crack size and relative voltage is independent of the shape of the distribution! On the other hand, it is not independent of the ratio of exponents γ/α. However, in the case of the classical theory of strength, $\gamma = 1$ and $\alpha = \frac{1}{2}$. The energy balance criterion yields $\gamma = 2$ and $\alpha = 1$. Thus in both cases $\gamma/\alpha = 2$. This similarity is not coincidental, but does have physical grounds, arising from classical fracture mechanics. Thus it appears that the universality of the crack size distribution in the case of exponential thresholds without lower cutoff does have physical grounds.

It would be of interest to repeat the above treatment in the case of a failure threshold distribution with a nonzero lower cutoff. The resulting crack appearance probability however is not an elementary function. It contains a hypergeometric function. It has been shown, using Maxwell-Boltzmann statistics, that in the vicinity of the atomistic scale, strength distribution does not have a nonzero lower cutoff [11]. However, there hardly are reasons to assume that this would apply to mesoscopic material elements in general.

6 Discussion

Within the dilute-crack approximation, discussing one critical crack results as only one length approximation for the critical crack (Eqs. (11) and (22)). This, combined with the approximation that the critical crack is long, results as critical stress proportional to the upper limit of the failure threshold distribution (Eqs. (10) and (21)). Furthermore, having quenched disorder in the failure thresholds, and the rule for stress enhancement at the crack tip being deterministic, the dilute-crack approximation results as a deterministic approximation for the critical applied voltage. Thus the dilute-crack approximation does not produce any strength distribution for the material, and the results of the present study cannot be compared with strength distributions arising from previous treatments, which often have been found to be of Weibull or double exponential form [1–3,12–18].

The dilute-crack approximation does produce a system size effect on the critical crack length, as well as on the critical nominal stress, and these results can be compared with previous studies. The present study always yields the critical crack length proportional to the logarithm of linear system size, and the critical applied voltage inversely proportional to the square root of the critical crack length. These features appear be rather general in the failure of disordered materials, the exponent of the critical crack length however depending on the load shearing geometry in the vicinity of failed elements [1,2,12,14,15,17–19]. It is further worth noting that the asymptotic size effect on strength at large sizes may be logarithmic, algebraic or nonexistent [20]. Some observations indicate that there may be an optimum system size which yields a minimum in failure probability [21].

The apparent universality in the crack size distribution, in the absence of lower cutoff of failure thresholds, is a result of the dilute-crack approximation. The dilute-crack approximation does appear as a rather effective, and in the mind of the author, also a rather reliable tool for discussing disordered sheet materials. Though it would be of interest to know how crack interactions might chance the picture. Another open question is the effect of disorder introduced in the conductivity (stiffness) which above was spatially invariant. Then, of course, fracture and damage of three-dimensional bodies would be a research field of significant practical importance.

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